

PRICE DYNAMICS IN FINANCIAL MARKETS: A KINETIC APPROACH

D. MALDARELLA* AND L. PARESCHI*

The use of kinetic modelling based on partial differential equations for the dynamics of stock price formation in financial markets is briefly reviewed. The importance of behavioral aspects in market booms and crashes and the role of agents' heterogeneity in emerging power laws for price distributions is emphasized and discussed.

Introduction

The recent events of the 2008 world's financial crisis and its uncontrolled effect propagated among the global economic system, has produced a deep rethink of some paradigm and fundamentals in economic modelling of financial markets¹. A big amount of efforts has been done in the understanding of stock's price dynamics, but also in the attempt to derive useful models for the risk estimation or price prediction. Nevertheless the need to find a compromise between the extraordinary complexity of the systems and the request of quite simplified models from which some basic information can be derived, represents a big challenge, and it is one of the main difficulties one have to deal with, in the construction of models.

Any reasonable model need to rely on some fundamental hypotheses and to rest on a theoretical framework, which should be able to provide some basic and universal principles, this is the way all the models arising from the physical world are build up. Unfortunately, this is not an easy task when we deal with economic and financial systems. Looking at stock market in particular, it is not obvious to understand which are the fundamental dynamics to be considered and which aspects can be neglected in order to derive the basic issues.

One of the most classical approach has been to consider the efficient market hypothesis^{12, 13}. It relies on the belief that securities markets are extremely efficient in reflecting information about individual stocks and about the stock market as a whole. When information arises, the news spread very quickly and are incorporated into the prices of securities without delay. Thus, neither technical analysis, which is the study of past stock prices in an attempt to predict future prices, nor even fundamental analysis, which is the analysis of financial information such as company earnings, asset values, etc., to help investors select undervalued stocks, would enable an investor to achieve returns greater than those that could be obtained by holding a randomly selected portfolio of individual stocks with comparable risk.

The efficient market hypothesis is associated with the idea of a random walk^{2, 7, 13}, which is widely used in the finance literature to characterize a price series where all subsequent price changes represent random departures from previous prices. The logic of the random walk idea is that if any information is immediately reflected in stock prices, then to-morrow's price change will reflect only tomorrow's news and will be independent of the price changes today. Thus, resulting price changes must be unpredictable and random.

Strongly linked to the market efficiency hypothesis, is the assumption of rational behavior among the traders. Rationality of traders can be basically reassumed in two main features. First, when they receive new information,

* Department of Mathematics & CMCS, University of Ferrara, via Machiavelli 35, Ferrara, Italy.

e-mail addresses: dario.maldarella@unife.it, lorenzo.pareschi@unife.it

agents update their beliefs by evaluating the probability of hypotheses accordingly to Bayes' law. Second, given their beliefs, agents make choices that are completely rational, in the sense that they arise from an optimization process of opportune subjective utility functions.

This traditional framework is appealingly simple, and it would be very satisfying if its predictions were confirmed in the data. Unfortunately, after years of efforts, it has become clear that basic facts about the stock market, the average returns and individual trading behavior are not easily understood in this framework³⁰.

By the beginning of the twenty-first century, the intellectual dominance of the efficient market hypothesis had become far less universal. Many financial economists and statisticians began to believe that stock prices are at least partially predictable. A new breed of economists emphasized psychological and behavioral elements of stock-price determination. The behavioral finance approach has emerged in response to the difficulties faced by the traditional paradigm^{14, 31, 32}. It relies in the fact that some financial phenomena can be better understood using models in which some agents are not fully rational. In some behavioral finance models, agents fail to update their beliefs correctly. In other models, agents apply Bayes' law properly but make choices that are questionable, in the sense that they are incompatible with the optimization of suitable utility functions.

A strong impact in the field of behavioral finance has been given by the introduction of the prospect theory by Kahneman and Tversky^{18, 19}. They present a critique of expected utility theory as a descriptive model of decision making under risk and develop an alternative model. Under prospect theory, value is assigned to gains and losses rather than to final assets and probabilities are replaced by decision weights. The theory which they confirmed by experiments predicts a distinctive fourfold pattern of risk attitudes: risk aversion for gains of moderate to high probability and losses of low probability, and risk seeking for gains of low probability and losses of moderate to high probability. Further development in this direction were done by De Bondt and Thaler¹⁰ who effectively form the start of what has become known as behavioral finance. They discovered that people systematically overreacting to unexpected and dramatic news events results in substantial weak-form inefficiencies in the stock market.

Recently, *agent based modelling* methods have given an important contribute and provided a huge quantity of numerical simulations^{17, 21, 23–25}. The idea is to produce a big mass of artificial data and to observe how they can fit

with empirical observations. This approach is now also supported by the availability of many recorded empirical data³³. The aim of the construction of such microscopic models of financial markets is to reproduce the observed statistical features of market movements (e.g. fat tailed return distributions, clustered volatility, cycles, crashes) by employing highly simplified models with large numbers of agents. Microscopic models of financial markets are highly idealized as compared to what they are meant to model³³. The relevant part of physics that is used to build such models of financial markets consists in methods from statistical mechanics. This attempt by physicists to map out the statistical properties of financial markets considered as complex systems is usually referred to as econophysics^{15, 28, 35}.

The need to recover mathematical models which can display such scaling properties, but also capable to deal with systems of many interacting agents and to take into account the effects of collective endogenous dynamics, put the question on the choices of the most appropriate mathematical framework to use. In fact, besides numerical simulations, it is of paramount importance to have a rigorous mathematical theory which permits to identify the essential features in the modelling originating the stylized facts. The classical framework of stochastic differential equations which played a major rule in financial mathematics seems inadequate to describe the dynamics of such systems of interacting agents and their emerging collective behavior.

In the last years a new approach based on the use of *kinetic and mean field models* and related mathematical tools has appeared in the mathematics and physics community^{3, 5, 8, 9, 11, 16, 20}. We refer to²⁹ for a recent review on the use of microscopic and kinetic modelling in socioeconomic sciences. Kinetic theory was introduced in order to give a statistical description of systems with many interacting particles. Rarefied gases can be thought as a paradigm of such complex systems, in which particles are described by random variables which represents their physical states, like position and velocity. A Boltzmann equation then prescribes the time evolution for the particles density probability function⁴. This seems to fit very well with the necessity to prescribe how the trading agents interacting in a stock market are leaded to form their expectations and reevaluate their choices on the basis of the influence placed on the neighbor agents' behavior rather than the flux of news coming from some fundamental analysis or direct observations of the market dynamic. The kinetic approach reveals particularly powerful when from some simple local interaction rules some global features

for the whole system has to be derived, but also in the study of asymptotic regimes and universal behaviors described by Fokker-Planck equations.

Here we briefly review some recent advances in this direction concerned with the kinetic modelling of the price dynamics in a simple stock market where two types of agents interact²⁶. Other kinetic and mean field approaches to price formation have been considered in^{8, 20}. Most of what we will present here has been inspired by the work of Lux and Marchesi^{23–25} on microscopic models for the stock market. Quite remarkably, however, behavioral features are taken into account in our model. In spite of its structural simplicity the kinetic model is able to reproduce many stylized facts such as lognormal and power laws price profiles and the appearance of market booms, crashes and cycles. As it is shown, non rational behavioral aspects and agents' heterogeneity are essential components in the model to achieve such behaviors.

Kinetic Modelling for Price Formation

Opinion modelling : The collective behavior of a system of trading agents can be described by introducing a state variable $y \in [-1, 1]$ and the relative density probability function $f(y)$ which, for each agent, represent respectively the propensity to invest and the probability to be in such a state. Positive values of y represent potential buyers, while negative values characterize potential sellers, close to $y = 0$ we have undecided agents. Clearly

$$\rho(t) = \int_{-1}^1 f(y, t) dy, \quad (1)$$

represents the number density. Moreover we define the mean investment propensity

$$Y(t) = \frac{1}{\rho(t)} \int_{-1}^1 f(y, t) y dy. \quad (2)$$

Traders are allowed to compare their strategies and to reevaluate them on the basis of a compromise opinion dynamic. This is done by assigning simple binary interaction rules, where, if the pair (y, y_*) and (y', y'_*) represent respectively the preinteraction and post-interaction opinions, we have

$$\begin{aligned} y' &= (1 - \alpha_1 H(y)) y + \alpha_1 H(y) y_* + D(y) \eta, \\ y'_* &= (1 - \alpha_1 H(y_*)) y_* + \alpha_1 H(y_*) y + D(y_*) \eta_*. \end{aligned} \quad (3)$$

Here $\alpha_i \in [0, 1]$ measures the importance the individuals place on others opinions in forming expectations

about future price changes. The random variables η and η_* are assumed distributed accordingly to $\Theta(\eta)$ with zero mean and variance σ^2 and measure individual deviations to the average behavior. The functions $H(y)$ and $D(y)$ characterize respectively, herding and diffusive behavior. Simple examples of herding function and diffusion function are given by

$$H(y) = a + b(1 - |y|), \quad D(y) = (1 - y^2)^\gamma,$$

with $0 \leq a + b \leq 1$, $a \geq 0$, $b > 0$, $\gamma > 0$. A kinetic model for opinion formation based on such interactions was recently introduced by Toscani³⁴.

Market influence : The traders are also influenced by the dynamics of stock market's price, so a coupling with the price dynamic has to be considered. With the same kinetic setting we define the probability density $V(s, t)$ of a given price s at time t . The market price $S(t)$ is then defined as the mean value

$$S(t) = \int_0^\infty V(s, t) s ds. \quad (4)$$

Price changes are modeled as endogenous responses of the market to imbalances between demand and supply characterized by the mean investment propensity accordingly to the following price adjustment

$$s' = s + \beta \rho Y(t) s + \eta s, \quad (5)$$

where $\beta > 0$ represents the price speed evaluation and η is a random variable with zero mean and variance ζ^2 distributed accordingly to $\Psi(\eta)$

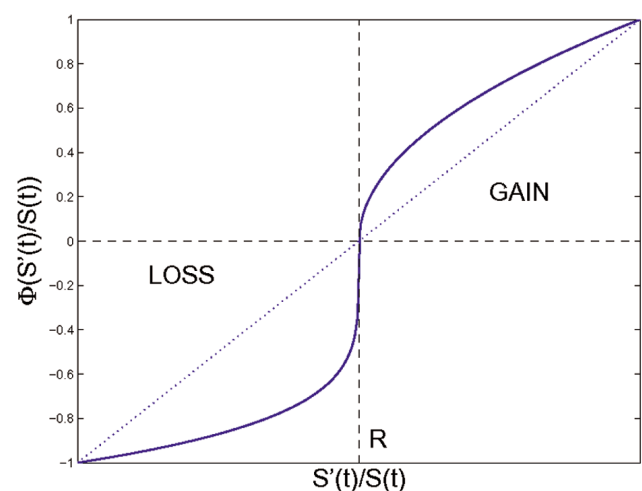


Fig 1: An hypothetical value function. The reference point R is the value of S'/S such that $\Phi(R) = 0$. The value function decision makers use to assess the possible shifts away from the reference point is concave in the domain of gains and convex in the domain of losses.

To take into account the influence of the price in the mechanism of opinion formation of traders, we introduce a normalized value function $\Phi = \Phi(\dot{S}(t)/S(t))$ in $[-1, 1]$ in the sense of Kahneman and Tversky^{18, 19} that models the reaction of individuals towards potential gain and losses in the market. Thus we reformulate the binary interaction rules in the following way

$$y' = (1 - \alpha_1 H(y) - \alpha_2)Y + \alpha_1 H(y)y_* + \alpha_2 \Phi + D(y)\eta, \quad (6)$$

$$y'_* = (1 - \alpha_1 H(y_*) - \alpha_2)y + \alpha_1 H(y_*)y_* + \alpha_2 \Phi + D(y_*)\eta_*.$$

Here $\alpha_1 \in [0, 1]$ and $\alpha_2 \in [0, 1]$, with $\alpha_1 + \alpha_2 \leq 1$, measure the importance the individuals place on others opinions and actual price trend in forming expectations about future price changes. This permits to introduce behavioral aspects in the market dynamic and to take into account the influence of psychology and emotivity on the behavior of the trading agents.

Note that agents influence the price through their mean propensity to invest $Y(t)$ and at the same time the price trend influences their mean propensity through the value function Φ . Thus, except for the particular shape of the value function, if the mean propensity is initially (sufficiently) positive then it will continue to grow together with the price and the opposite occurs if it is initially (sufficiently) negative. The market goes towards a boom (exponential growth of the price) or a crash (exponential decay of the price).

Lognormal behavior : A set of Boltzmann equations for the evolution of the unknown densities $f(y, t)$ and $V(s, t)$ can be obtained using the standard tools of kinetic theory⁴. Such system reads

$$\frac{\partial f}{\partial t} = Q(f, f), \quad (2.7)$$

$$\frac{\partial V}{\partial t} = L(V),$$

where the quadratic operator Q and the linear operator L can be conveniently written in weak form as

$$\begin{aligned} \int_{-1}^1 Q(f, f) \varphi(y) dy &= \\ \int_{[-1,1]^2} \int_{R^2} B(y, y_*) f(y) f(y_*) (\varphi(y') - \varphi(y)) d\eta d\eta_* dy_* dy, \\ \int_0^\infty L(V) \varphi(s) ds &= \int_0^\infty \int_R b(s) V(s) (\varphi(s') - \varphi(s)) d\eta ds. \end{aligned}$$

In the above equations φ is a test function and the transition rates have the form

$$B(y, y_*) = \Theta(\eta) \Theta(\eta_*) \chi(|y'| \leq 1) \chi(|y'_*| \leq 1),$$

$$b(s) = \Psi(\eta) \chi(s' \geq 0),$$

with $\chi(\cdot)$ the indicator function.

A simplified Fokker-Planck model which preserves the main features of the original Boltzmann model is obtained under a suitable scaling of the system. In such scaling all agents interact simultaneously with very small variations of their investment propensity (see²⁶ for details). This allows us to recover the following Fokker-Planck system

$$\begin{cases} \frac{\partial f}{\partial t} + \frac{\partial}{\partial y} [(\rho \alpha_1 H(y)(Y - y) + \rho \alpha_2 (\Phi - y)) f] \\ = \frac{\partial^2 \rho}{2} \frac{\partial^2}{\partial y^2} [D^2(y) f], \\ \frac{\partial V}{\partial t} + \frac{\partial}{\partial s} (\beta \rho Y s V) = \frac{\zeta^2}{2} \frac{\partial^2}{\partial s^2} (s^2 V), \end{cases} \quad (8)$$

where we kept the original notations for all the scaled quantities.

The above equation for the price admits the self similar lognormal solution^{8,26}

$$V(s, t) = \frac{1}{s (2 \log(Z(t)^2) \pi)^{\frac{1}{2}}} \exp \left(-\frac{(\log(sZ(t)))^2}{2 \log(Z(t)^2)} \right), \quad (9)$$

where $Z(t) = \sqrt{E(t)}/S(t)$ and $E(t)$ satisfies the differential equation

$$\frac{dE}{dt} = (2\beta Y + \zeta^2) E.$$

Playing a Different Strategy

We consider now in the stock market the presence of traders who deviate their strategy from the mass. We introduce trading agents who rely in a fundamental value for the traded security. They are buyer while the price is below the fundamental value and seller while the price is above. Expected gains or losses are then evaluated from deviations of the actual market price and just realized only whether or not the price will revert towards the fundamental value. Such agents are not influenced by other agents' opinions.

The microscopic interactions rules for the price formation now reads

$$s' = s + \beta(\rho_Y(t)s + \rho_F\gamma(S_F - s)) + \eta s, \quad (10)$$

where S_F represent the fundamental value of the price, ρ_f is the number density for such trading agents performing a different strategy, while γ is the reaction strength to deviations from the fundamental value. If we are interested in steady states we can ignore the possibility of a strategy exchange between traders and the resulting kinetic system has the same structure (8). We refer to²⁶ for a complete treatment of a model including strategy exchanges.

Equilibrium states : The system of equations (8) in the simplified case of H constant admits the following possible macroscopic equilibrium configurations²⁶

- (i) $\rho_F \neq 0, S = S_F, Y = 0, \Phi(0) = 0,$
- (ii) $\rho_F = 0, Y = 0, \Phi(0) = 0, S$ arbitrary,
- (iii) $\rho_F = 0, Y = Y_*,$ with $Y_* = \Phi(\beta t_c Y_*), S = 0,$

where only configuration (i) takes into account the presence of both types of traders. Note, however, that if the reference point for the value function is different from zero, namely $\Phi(0) \neq 0$, configuration (i) and (ii) are not possible. This is in good agreement with the fact that an emotional perception of the market acts as a source of instability for the market itself. In contrast configuration (iii), corresponding to a market crash, can be achieved also for $\Phi(0) \neq 0$.

Emergence of power laws : The presence of fundamentalists leads to the following Fokker-Planck equation for the probability density function V

$$\frac{\partial V}{\partial t} + \frac{\partial}{\partial s} [\beta(\rho_Y s + \rho_F \gamma(S_F - s))V] = \frac{\zeta^2}{2} \frac{\partial^2}{\partial s^2} (s^2 V) \quad (11)$$

If we consider the equilibrium configuration (i) a steady state for the Fokker-Planck equation can be computed in the form of a Gamma distribution^{3,9,26}

$$V^\infty(s) = C_1(\mu) \frac{1}{s^{1+\mu}} e^{-\frac{(\mu-1)S_F}{s}}, \quad (12)$$

where $\mu = 1 + 2\rho_F\gamma/\zeta^2$ and $C_1(\mu) = ((\mu-1)S_F)^\mu / \Gamma(\mu)$ with $\Gamma(\cdot)$ being the usual Gamma function. Therefore the price distribution exhibits a Pareto tail behavior. $\square$

References

1. F. G. Adams, The world financial crisis: new economy, globalisation and old-fashioned philosophy. *World Economics*, **10**(1), 4558, (2009).
2. L. Bachelier, Theorie de la speculation, *Annales Scientifiques de l'cole Normale Suprieure* **3** (17), 21-86, (1900).
3. J.-P. Bouchaud and M. Mézard, Wealth condensation in a simple model of economy, *Physica A*, **282**, 536, (2000).
4. C. Cercignani, *The Boltzmann equation and its applications*, Springer Series in Applied Mathematical Sciences, 67 (Springer-Verlag) (1988).
5. A. Chatterjee, B.K. Chakrabarti and R.B. Stinchcombe, Master equation for a kinetic model of trading market and its analytic solution. *Phys. Rev. E* **72**, 026126 (2005).
6. R. Cont and J.-P. Bouchaud, Herd behavior and aggregate fluctuations in financial markets, *Macroeconomic dynamics*, **4**, 170-196 (2000).
7. P. H. Cootner, *The random character of stock market prices*. MIT Press, (1964).
8. S. Cordier, L. Pareschi and C. Piatecki, Mesoscopic modelling of financial markets. *J. Stat. Phys.*, **134**, 161-184, (2009).
9. S. Cordier, L. Pareschi and G. Toscani, On a kinetic model for a simple market economy. *J. Stat. Phys.*, **120**, 253-277, (2005).
10. W.F.M., De Bondt and R. Thaler, Does the stock market overreact? *The Journal of Finance*, 793-805, (1985).
11. B. Düring, D. Matthes and G. Toscani, Kinetic Equations modelling Wealth Redistribution: A comparison of Approaches. *Phys. Rev. E* **78**, 056103 (2008).
12. E.F. Fama, Efficient Capital Markets: a Review of Theory and Empirical Work. *Journal of Finance*, **25** 383-417 (1970).
13. E. F. Fama, Random Walks in Stock Market Prices. *Financial Analysts Journal*, **21** 55-59 (1965).
14. E. Fama, Market efficiency, long-term returns, and behavioral finance. *Journal of Financial Economics*, **49** 283-306 (1998).
15. X. Gabaix, P. Gopikrishnan, V. Plerou and H. E. Stanley, A Theory of Power-Law Distributions in Financial Market Fluctuations, *Nature* 423, 267-270, (2003).
16. S. Ispolatov, P. L. Krapivsky and S. Redner, Wealth distributions in asset exchange models. *Eur. Phys. Jour. B* **2**, 267-276, (1998).
17. G. Kim and H. M. Markowitz, Investments rules, margin and market volatility, *Journal of portfolio management*, 16, (1989), 45-52.
18. D. Kahneman and A. Tversky, Prospect Theory: An Analysis of Decision under Risk, *Econometrica*, 263-292, (1979).
19. D. Kahneman and A. Tversky, *Choices, Values, and Frames*. Cambridge: Cambridge University Press, (2000).
20. J.-M. Lasry and P.-L. Lions, Mean field games, *Japanese Journal of Mathematics*, **2**, 229-260, (2007).
21. M. Levy, H. Levy and S. Solomon, A microscopic model of the stock market: Cycles, booms, and crashes. *Economics Letters*, **45** 103-111, (1994).
22. M. Levy and S. Solomon, Dynamical explanation for the emergence of power law in a stock market. *International Journal of Modern Physics C*, **7**, 65-72, (1996).
23. T. Lux, The socio-economic dynamics of speculative markets: interacting agents, chaos, and the fat tails of return distributions, *Journal of Economic Behavior & Organization* Vol. **33**, 143-165, (1998).

24. T. Lux and M. Marchesi, Volatility clustering in financial markets: a microscopic simulation of interacting agents, *International Journal of Theoretical and Applied Finance.*, **3**, 675–702, (2000)
25. T. Lux and M. Marchesi, Scaling and criticality in a stochastic multi-agent model of a financial market. *Nature* **397** No. 11, 498–500, (1999).
26. D. Maldarella and L. Pareschi, Kinetic models for socioeconomic dynamics of speculative markets. *CMCS Report*, (2009).
27. P. Milgrom and N. Stokey, Information, trade and common knowledge. *Journal of economic theory*, **26** 17–27 (1982)
28. R.N. Mantegna and H. E. Stanley, Scaling Behaviour in the dynamic of an Economic index *Nature*, 46–48, (1995).
29. G. Naldi, L. Pareschi and G. Toscani, *Mathematical modelling of collective behavior in socio-economic and life sciences*, Birkhauser, (2010).
30. R. Rosenberg and R. Lanstein, Persuasive evidence of market inefficiency. *Journal of Portfolio Management*, 13 (1985) 9–17
31. R. Shiller, Do stock prices move too much to be justified by subsequent changes in dividends? *American Economic Review*, **71** 421–436 (1981)
32. Shiller, R., Stock prices and social dynamics. *Brookings Papers on Economic Activity*, **2** 457–498 (1984)
33. E. Samanidou, E. Zschischang, D. Stauffer and T. Lux, Agentbased models of financial markets. *Report on progress in physics*, **70** 409–450 (2007).
34. G. Toscani, Kinetic Models of opinion formation, *Comm. Math. Scie.*, (2006).
35. J. Voit, *The Statistical Mechanics of Financial Markets*, Springer, (2005).